6.2 Permutation Group, Even and odd Permutations, Alternating group.

~~6.1) Permutation Group:-~~

Q.① Define Permutation group.

Ans:- If $f: S \to S$ and f is one-one onto, then f is a permutation of degree n.

OR

Let S is a finite set having n distinct elements. Then a one-one mapping of S ~~into~~ onto itself is called a permutation of set S, of degree n.

Let $S = \{a_1, a_2, a_3, \dots, a_n\}$ be finite set having n distinct elements.

Let $f(a_1) = b_1,\ f(a_2) = b_2,\ f(a_3) = b_3; \dots f(a_n) = b_n$

where $\{b_1, b_2, b_3, \dots, b_n\} = \{a_1, a_2, a_3, \dots, a_n\}$

i.e. $b_1, b_2, b_3, \dots, b_n$ is nothing but some arrangement of elements $a_1, a_2, a_3, \dots, a_n$ of S.

$$\therefore f = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix}$$

Ex① $S = \{1, 2, 3, 4\}$

$f(1) = 2,\ f(2) = 4,\ f(3) = 1,\ f(4) = 3$

Then, $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

Similarly, $g: S \to S$, such that $g(1)=1,\ g(2)=3,\ g(3)=2,\ g(4)=4$

then, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Here f and g are permutations of degree 4

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6.2.2 Equality of two Permutations:—

Two Permutations f and g of degree n are said to be equal if we have $f(a) = g(a)$, $\forall a \in S$

For Example,

if $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $g = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

are two permutation of set $S = \{1, 2, 3, 4\}$

Now, $f(1) = 2$, $f(2) = 4$ | $g(2) = 4$, $g(4) = 3$
$f(3) = 1$, $f(4) = 3$ | $g(1) = 2$, $g(3) = 1$

Hence, $f = g$.

6.2.3 Total number of distinct Permutations of degree n. :—

Let S be a finite set having n distinct elements. Then, $^nP_n = n!$ distinct arrangements of the elements of S.

Ex ① $S = \{1, 2, 3\}$ Then total no of Permutations of $S = 3! = 6$

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $T_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$T_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $T_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $T_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

6.2.4 Identity Permutation :—

If I is a permutation of degree n such that I replaces each element by the element itself, I is called the identity permutation of degree n.

For example, $I = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$ is the identity permutation of degree n.

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6.2.5 Product or Composite of two permutations :-

The Product or Composite of two Permutations f and g of degree n, denoted by fg is obtained by first carrying out the operation defined by f and then by g.

Suppose P_n is the set of all permutation of degree n and let $f, g \in P_n$ defined by

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \text{ and } g = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

$$fg = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}\begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

obviously, fg is a permutation of degree n

Hence, $f, g \in P_n \Rightarrow fg \in P_n$

Ex ① Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ then find fg and gf

Soln:- $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$ —— (i)

and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ —— (ii)

Now,

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Similarly

$$gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 & 1 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

obviously, $fg \neq gf$.

Group of Permutations {or Symmetric group S_n}

Theorem:- The set P_n of all permutation on n symbols is a finite group of order $n!$ with respect to composite of mapping as the operation. For $n \leq 2$, this group is Abelian and for $n > 2$, it is non-Abelian.

Proof:- Do yourself

Even and odd Permutation:-

A permutation is said to be even if it can be re-written as a product of an even number of transposition, otherwise it is said to be an odd permutation.

Ex ① Examine whether the permutation $\{1\ 2\ 3\}\{4\ 6\ 5\}$ is even or odd.

Soln:- We have.

$$\{1, 2, 3\} = \{1, 2\}\{1, 3\}$$

and $\{4, 5, 6\} = \{4, 6\}\{4, 5\}$

$$\therefore \{1\ 2\ 3\}\{4\ 5\ 6\} = \{1\ 2\}\ \{1\ 3\}\{4\ 6\}\{4\ 5\}$$

Since the number of transpositions is even

$\therefore$ the permutation

Ex ② Determine whether $f = (1\ 2\ 3\ 4\ 5)(1\ 2\ 3)(4\ 5)$ is an even function.

Soln:- We can write f as

$$f = (1\ 2)(1\ 3)(1\ 4)(1\ 5)(1\ 2)(1\ 3)(4\ 5)$$

Since the number of transposition is odd

Therefore, f is an odd permutation.